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Deformed Supersymmetric Oscillators

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We construct and discuss the Fock-space representation and the number operator for a deformed supersymmetric oscillator with "peculiar" statistics. We suggest a possible generalization to multimode deformed oscillators.

1. INTRODUCTION

The subject of quantum statistics, which is different from ordinary Bose and Fermi statistics, has attracted much attention in the past few years. One motivation comes from the study of some phenomena in condensed matter where the dynamics is essentially two-dimensional, thus allowing anyon-like statistics (Leinaas and Myrheim, 1977; Wilczek, 1982a,b; Wu, 1984a,b). Another motivation comes from the theoretical and experimental search for possible violation of the Pauli exclusion principle in four dimensions (Ignatiev and Kuzmin, 1987; Mohapatra and Greenberg, 1989; Miljanić *et al.*, 1990; Ramberg and Snow, 1990), where quon-like statistics (Greenberg, 1990, 1991; Meljanac and Perica, 1994) might play a significant role. In either case, quantum groups and algebras (Drinfeld, 1986; Jimbo, 1985) have offered a new insight into the subject. The introduction of q-deformations of the Heisenberg–Weyl algebras has led to the investigation of particles interpolating between bosons and fermions (Biedenharn, 1989; Macfarlane, 1989). The q-bosons have been introduced and discussed in a variety of ways (Tuszynski *et al.*, 1993; Isakov, 1993; Meljanac *et al.*, 1994a,b; Bardek *et al.*, 1994a,b; Bonatsos and Daskaloyannis, 1992; Odaka *et al.*, 1991). Particularly useful formulations of associative q-boson algebras are proposed through the Yang–

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Baxter R-matrix (Kulish, 1991; Fairlie and Zachos, 1991; Meljanac *et al.*, 1994a,b), which generalizes the notion of permutational symmetry. The simplest such algebras, associated to 4×4 R-matrices, were investigated to some extent in Van der Jeugt (1993) and three types of deformed algebras were found, among them a "peculiar" algebra which corresponded to the R-matrix of the eight-vertex form. The detailed structure of the "peculiar" algebra was not discussed in Van der Jeugt (1993) and it remains unclear whether this algebra makes sense physically, i.e., whether there exists a well-defined Fock-space representation with positive norms and number operators.

In this paper we investigate the structure of this "peculiar" algebra. We construct and discuss the corresponding Fock-space representation and show that norms of all states are positive definite (Section 2). We construct and discuss the number operators and investigate the origin of "peculiarity." We show that this algebra represents a new kind of deformed supersymmetric oscillator (Section 3). We suggest a possible generalization of this "peculiar" algebra to an arbitrary number of oscillators with the corresponding R-matrix (Section 4). Section 5 is devoted to our conclusion.

2. FOCK-SPACE REPRESENTATION OF 'PECULIAR' ALGEBRA

We start with the following R-matrix formulation of the q-deformed boson algebra of the operators a_i, $a_i^\dagger$, $i = 1, \ldots, n$:

$$a_i a_j - p R_{ij,kl} a_l a_k = 0 \tag{1}$$
$$a_i a_j^\dagger - p' R_{ki,jl} a_k^\dagger a_l = \delta_{ij}$$

with the summation over repeated indices and $a_i^\dagger$ is the Hermitian conjugate of a_i. Hermiticity implies

$$p' R_{ij,kl} = p'^* R^*_{lk,ji} \tag{2}$$

or $\hat{R}^\dagger = \hat{R}$, where $\hat{R}_{ij,kl} = P_{ij,mn} R_{mn,kl}$ and P is the permutation operator $P^2 = P$, $P_{ij,kl} = \delta_{jk}\delta_{il}$. Associativity implies:

(A) The Yang–Baxter equation

$$\sum_{u,v,w} R_{ab,uv} R_{vw,cd} R_{ue,fw} = \sum_{u,v,w} R_{be,uv} R_{wu,fc} R_{av,wd} \tag{3}$$

(B) The Hecke condition

$$(p\hat{R} - 1)(p'\hat{R} + 1) = 0, \qquad \hat{R} = PR \tag{4}$$

The solutions of equations (3) and (4) for $n = 2$ are given in Van der Jeugt (1993). A complete list of solutions of the Yang–Baxter equation (3) for n

= 2 is given in Hietarinta (1991). Among them there is a solution with "peculiar" statistics associated with the R-matrix of the eight-vertex form

$$pR = \begin{bmatrix} \frac{1-q^2}{2} + \epsilon q & 0 & 0 & \epsilon' \frac{1-q^2}{2} \\ 0 & \epsilon'' \frac{1+q^2}{2} & \frac{1-q^2}{2} & 0 \\ 0 & \frac{1-q^2}{2} & \epsilon'' \frac{1+q^2}{2} & 0 \\ \epsilon' \frac{1-q^2}{2} & 0 & 0 & \frac{1-q^2}{2} - \epsilon q \end{bmatrix} \tag{5}$$

The corresponding "peculiar" oscillator algebra is obtained from (1) with $p'R = q^{-2}pR$:

$$\begin{aligned}
(1 - \epsilon q)a_1^2 &= \epsilon'(1 + \epsilon q)a_2^2 \\
a_1 a_2 &= \epsilon'' a_2 a_1 \\
a_1 a_1^\dagger &= 1 + \left(\epsilon q^{-1} + \frac{q^{-2} - 1}{2}\right) a_1^\dagger a_1 + \left(\frac{q^{-2} - 1}{2}\right) a_2^\dagger a_2 \\
a_2 a_2^\dagger &= 1 + \left(-\epsilon q^{-1} + \frac{q^{-2} - 1}{2}\right) a_2^\dagger a_2 + \left(\frac{q^{-2} - 1}{2}\right) a_1^\dagger a_1 \\
a_1 a_2^\dagger &= \left(\epsilon'' \frac{1 + q^{-2}}{2}\right) a_2^\dagger a_1 + \left(\epsilon' \frac{q^{-2} - 1}{2}\right) a_1^\dagger a_2 \\
a_2 a_1^\dagger &= \left(\epsilon'' \frac{1 + q^{-2}}{2}\right) a_1^\dagger a_2 + \left(\epsilon' \frac{q^{-2} - 1}{2}\right) a_2^\dagger a_1
\end{aligned} \tag{6}$$

where $q \in \mathbf{R}$, $\epsilon^2 = \epsilon'^2 = \epsilon''^2 = 1$. When $q^2 = 1$, the above algebra (6) represents one Bose and one Fermi oscillator which commute or anticommute (depending on whether ϵ'' is 1 or -1). We observe that the "peculiar" algebra (6) has no well-defined number operators N_1, N_2 in the usual sense: $[N_i, a_j] = -a_i\delta_{ij}$, $[N_i, a_j^\dagger] = \alpha_i\delta_{ij}$, $i, j = 1, 2$. From $[N_1, a_1] = -a_1$ it follows that $[N_1, a_1^2] = -2a_1^2$. Owing to (6) one obtains $[N_1, a_2^2] = -2a_2^2$, which contradicts the demanded relation $[N_1, a_2] = 0$. Hence the relations $[N_1, a_2] = [N_2, a_1] = 0$ contradict (6). However, the total number operator $N = N_1 + N_2$ is well defined. Of course, when $q^2 = 1$, the number operators N_1 and N_2 are also well defined, i.e., $N_{1,2} = N_{\mathrm{B,F}}$.

Let us assume that there is a vacuum $|0, 0\rangle$ satisfying $a_i|0, 0\rangle = 0$, $i = 1, 2$. The excited states can be constructed by multiple action of the operators $a_1^\dagger$ and $a_2^\dagger$ on the vacuum $|0, 0\rangle$ and are of the form

$$|n_1, n_2\rangle \propto (a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}|0, 0\rangle, \qquad n_1, n_2 \in \mathbf{N} \tag{7}$$

Notice that the action of N_1 (N_2) on the states (7) is not well defined for $n_2 \geq 2$ ($n_1 \geq 2$) and hence, in general, n_1 (n_2) is not an eigenvalue of N_1 (N_2). This is a consequence of the quadratic relation $a_1^2 \propto a_2^2$ [equations (6)] for $q^2 \neq 1$. Furthermore, we find that the $|n_1, n_2\rangle$ states are degenerate (linearly dependent) for the fixed sum $n_1 + n_2 = n$, in the following sense: $|n_1, n_2\rangle \propto |n_1 - 2k, n_2 + 2k\rangle$, $k \in \mathbf{Z}$, $n_1 - 2k \geq 0$, $n_2 + 2k \geq 0$, and $|n_1 + 1, n_2 - 1\rangle \propto |n_1 + 1 - 2k, n_2 - 1 + 2k\rangle$, $k \in \mathbf{Z}$; $n_1 + 1 - 2k \geq 0$, $n_2 - 1 + 2k \geq 0$. The states for fixed n can be reduced to two states, $|n, 0\rangle$ and $|(n - 1), 1\rangle$, or, alternatively, to $|0, n\rangle$ and $|1, (n - 1)\rangle$. Hence, the complete set of states can be represented by two symmetric pictures (for $q^2 \neq 1$)

$$|n, \nu\rangle \propto (a_1^\dagger)^n(a_2^\dagger)^\nu|0, 0\rangle \tag{8a}$$

$$|\nu, n\rangle \propto (a_1^\dagger)^\nu(a_2^\dagger)^n|0, 0\rangle \tag{8b}$$

where $n \in \mathbf{N}_0$, $\nu = 0, 1$. Now, n (ν) is the eigenvalue of N_1 (N_2) in the picture (8a), and of N_2 (N_1) in the picture (8b). In the following we use the first picture (8a).

There are two towers of states generated by the $a_1^\dagger$ creation operator. One tower is $|n, 0\rangle$, generated from the $|0, 0\rangle$ vacuum ($\nu = 0$). The other tower is $|n, 1\rangle$, generated from the second vacuum $|0, 1\rangle$ ($\nu = 1$). Using the algebra (6), we find that

$$a_1^\dagger a_1(a_1^\dagger)^n(a_2^\dagger)^\nu|0, 0\rangle = \phi_1(n, \nu)(a_1^\dagger)^n(a_2^\dagger)^\nu|0, 0\rangle$$

$$\phi_1(n, \nu) = \frac{1}{2}[n]_{(\epsilon q)^{-1}}(1 + (\epsilon q)^{1-n-2\nu}) \tag{9}$$

where

$$[n]_{(\epsilon q)^{-1}} = \frac{(\epsilon q)^{-n} - 1}{(\epsilon q)^{-1} - 1}, \qquad n \in \mathbf{N}_0, \qquad \nu = 0, 1$$

It is important to observe that, for $q \in \mathbf{R}$, the function ϕ_1 is positive: $\phi_1(n, \nu) > 0$, $\forall n \in \mathbf{N}_0$, $\nu = 0, 1$. Furthermore, $\phi(n, \nu)$ cannot be written as a function of one variable. If this could have been done, this would mean that there would be only one tower of states, and that $a_1 \propto a_2$. Hence, all states $(a_1^\dagger)^n(a_2^\dagger)^\nu|0, 0\rangle$, picture (8a), have positive-definite norms and can be normalized. The normalized states are

$$|n, \nu\rangle = \frac{(a_1^\dagger)^n (a_2^\dagger)^\nu |0, 0\rangle}{\sqrt{[\phi_1(n, \nu)]!}} \tag{10}$$

where $[\phi_1(n, \nu)]! = \phi_1(n, \nu) \cdots \phi_1(1, \nu)$, $[\phi_1(0, \nu)]! = 1$, and the orthonormality condition reads $\langle n, \nu | n', \nu'\rangle = \delta_{nn'}\delta_{\nu\nu'}$, $\nu, \nu' = 0, 1$. Owing to this orthonormality relation, any linear combination of states, Eq. (10), has a positive norm. In particular,

$$\|\alpha | n, 0\rangle + \beta | n - 1, 1\rangle\|^2 = |\alpha|^2 + |\beta|^2 > 0$$

It is easy to find the action of the a_i, $a_i^\dagger$ operators on the states, Eq. (10), namely

$$\begin{aligned}
a_1^\dagger | n, \nu\rangle &= \sqrt{\phi_1(n + 1, \nu)}\, | n + 1, \nu\rangle \\
a_1 | n, \nu\rangle &= \sqrt{\phi_1(n, \nu)}\, | n - 1, \nu\rangle \\
a_2^\dagger | n, \nu\rangle &= \sqrt{\frac{[\phi_1((n + 2\nu), (1 - \nu))]!}{[\phi_1(n, \nu)]!}} \\
&\quad \times \left(\frac{1 - \epsilon q}{\epsilon'(1 + \epsilon q)}\right)^\nu (\epsilon'')^n | (n + 2\nu), (1 - \nu)\rangle \\
a_2 | n, \nu\rangle &= \sqrt{\frac{[\phi_1(n, \nu)]!}{[\phi_1((n - 2 + 2\nu), (1 - \nu))]!}} \\
&\quad \times \left(\frac{1 - \epsilon q}{\epsilon'(1 + \epsilon q)}\right)^{1-\nu} (\epsilon'')^n | (n - 2 + 2\nu), (1 - \nu)\rangle
\end{aligned} \tag{11}$$

In the picture (8a), the $a_1^\dagger$ operator builds two infinite towers on $|0, 0\rangle$ and $|0, 1\rangle$, respectively, whereas the a_2, $a_2^\dagger$ operators interconnect the two towers. In the picture (8b), in which the indices are interchanged, $1 \leftrightarrow 2$ and $\epsilon \leftrightarrow -\epsilon$, the $a_2^\dagger$ operator creates two towers based on $|0, 0\rangle$ and $|0, 1\rangle$, while the $a_1^\dagger$ operator braids between these two towers. Equations (10)–(11) hold with $1 \leftrightarrow 2$, $\epsilon \leftrightarrow -\epsilon$.

The number operator N_1 counts the $a_1^\dagger$ excitations, and can be written as

$$\begin{aligned}
N_1 &= \sum_{n=1}^{\infty} [c_1(n)(a_1^\dagger)^n a_1^n + c_2(n) a_2^\dagger (a_1^\dagger)^{n-1} a_1^{n-1} a_2] \\
&= a_1^\dagger a_1 + \frac{(\epsilon q^{-1} - 1)(\epsilon q^{-1} + 3)}{(\epsilon q^{-1} + 1)^2} (a_1^\dagger)^2 a_1^2 + \frac{1 - q^{-2}}{1 + q^{-2}} a_2^\dagger a_1^\dagger a_1 a_2 + \cdots \\
&= a_1^\dagger a_1 + \frac{\epsilon q^{-1} + 3}{\epsilon q^{-1} - 1} (a_2^\dagger)^2 a_2^2 + \frac{1 - q^{-2}}{1 + q^{-2}} a_2^\dagger a_1^\dagger a_1 a_2 + \cdots
\end{aligned} \tag{12}$$

Note that $N_1 = a_1^\dagger a_1$ when $q^2 = 1$. Alternatively, we can write $a_1^\dagger a_1 = \phi_1(N_1, \nu) = \varphi_{1,\nu}(N_1)$ and $N_1 = \varphi_{1,\nu}^{-1}(a_1^\dagger a_1)$, $\nu = 0, 1$. The total number operator is

$$N = N_1 + \nu, \qquad \nu = 0, 1$$

3. DEFORMED SUSY OSCILLATORS

We can define the operators $Q_{ij} = a_i a_j^\dagger$ ($Q_{ij}^\dagger = Q_{ji}$) and $\tilde{Q}_{ij} = a_i^\dagger a_j$ ($\tilde{Q}_{ij}^\dagger = \tilde{Q}_{ji}$), $i, j = 1, 2$, satisfying [in the picture (8a)]

$$\begin{aligned}
Q_{ij} &= \delta_{ij} + p' R_{ki,jl} \tilde{Q}_{kl} \\
[N, Q_{ij}] &= [N, \tilde{Q}_{ij}] = 0, \qquad \forall i, j = 1, 2 \\
Q_{11} | n, \nu\rangle &= \phi_1(n + 1, \nu) | n, \nu\rangle \\
Q_{22} | n, \nu\rangle &= \phi_2(n + 2\nu, 1 - \nu) | n, \nu\rangle \\
Q_{12} | n, \nu\rangle &= \psi_{12}(n, \nu) | n - 1 + 2\nu, 1 - \nu\rangle \\
Q_{21} | n, \nu\rangle &= \psi_{21}(n, \nu) | n - 1 + 2\nu, 1 - \nu\rangle \\
Q_{12}^\dagger Q_{12} &= Q_{21} Q_{12} = \psi_{12}^2(n, \nu) \\
Q_{12}^2 &= \psi_{12}(n, \nu)\psi_{12}(n - 1 + 2\nu, 1 - \nu)
\end{aligned} \tag{13}$$

where

$$\begin{aligned}
\phi_2(n, \nu) &= \frac{[\phi_1(n, \nu)]!}{[\phi_1(n - 2 + 2\nu, 1 - \nu)]!} \left(\frac{1 - \epsilon q}{\epsilon'(1 + \epsilon q)}\right)^{2(1-\nu)} \\
\psi_{12}(n, \nu) &= \sqrt{\frac{\phi_1(n - 2 + 2\nu, 1 - \nu)[\phi_1(n + 2\nu, 1 - \nu)]!}{[\phi_1(n, \nu)]!}} \\
&\quad \times \left(\frac{1 - \epsilon q}{\epsilon'(1 + \epsilon q)}\right)^{\nu} (\epsilon'')^n \\
\psi_{21}(n, \nu) &= \psi_{12}(n - 1 + 2\nu, 1 - \nu)
\end{aligned} \tag{14}$$

Analogous relations can be obtained for the operators $\tilde{Q}_{ij}$ and $\tilde{Q}_{ij}^\dagger$ using Eqs. (11) and (13). Notice that $(Q_{12})^2 \neq 0$ $[(\tilde{Q}_{12})^2 \neq 0]$ when $q^2 \neq 1$ and $(Q_{12})^2 = 0$ $[(\tilde{Q}_{12})^2 = 0]$ when $q^2 = 1$.

We can define the Hamiltonian H through

$$\begin{aligned}
&\{Q_{12}, Q_{12}^\dagger\} = 2H \\
&[H, Q_{12}] = [H, Q_{12}^\dagger] = [H, N] = 0 \\
&H | n, \nu\rangle = \tfrac{1}{2}((\psi_{12}(n, \nu))^2 + (\psi_{12}(n - 1 + 2\nu, 1 - \nu))^2) | n, \nu\rangle
\end{aligned} \tag{15}$$

and similarly the Hamiltonian $\tilde{H}$ through

$$
\begin{aligned}
\{\tilde{Q}_{12}, \tilde{Q}_{12}^{\dagger}\} &= 2\tilde{H} \\
[\tilde{H}, \tilde{Q}_{12}] &= [\tilde{H}, \tilde{Q}_{12}^{\dagger}] = [\tilde{H}, N] = 0 \\
\tilde{H}|n, \nu\rangle &= \tfrac{1}{2}((\tilde{\psi}_{12}(n, \nu))^2 + (\tilde{\psi}_{12}(n - 1 + 2\nu, 1 - \nu))^2)|n, \nu\rangle
\end{aligned}
\tag{16}
$$

The relations (15) and (16) define a new kind of q-deformation of the supersymmetric (SUSY) oscillator (Parthasarathy and Wiswanathan, 1991; Chung, 1995). We point out that the spectrum of H ($\tilde{H}$) is positive and degenerate, i.e., the states $|n, 0\rangle$ and $|n - 1, 1\rangle$ have the same energy $\frac{1}{2}((\psi(n, 0))^2 + (\psi(n - 1, 1))^2)$. These properties are typical for the SUSY oscillator (de Crombrugghe and Rittenberg, 1983; Gendensthein and Krive, 1985), except that for $q^2 \neq 1$ the energy levels are not equidistant. In the limit $q = +1$ the state $|n, 0\rangle$ ($|n - 1, 1\rangle$) is bosonic (fermionic) in the picture (8a). In the limit $q = -1$ the state $|0, n\rangle(|1, n - 1\rangle)$ is bosonic (fermionic) in the picture (8b).

The q-deformed SUSY algebra (15) is generated by the set $\{N, Q_{12}, Q_{12}^{\dagger}, H\}$ and the q-deformed SUSY algebra (16) by the set $\{N, \tilde{Q}_{12}, \tilde{Q}_{12}^{\dagger}, \tilde{H}\}$. Notice that our Hamiltonian H (and $\tilde{H}$) is invariant under the q-superalgebra since H and Q ($\tilde{H}$ and $\tilde{Q}$) mutually commute, in contrast to the Hamiltonian of the form $H = \{Q_+, Q_-\}$ mentioned in Parthasarathy and Wiswanathan (1991). The q-deformed supercharges, operators Q_{ij}, $\tilde{Q}_{ij}$, $i \neq j$, also braid between the two towers and preserve the total number operator $N = N_1 + \nu$. Although the operators Q and $\tilde{Q}$ are not nilpotent ($Q_{12}^2 \neq 0$ for $q^2 \neq 1$, contrary to the ordinary SUSY oscillator), their irreducible representations remain two-dimensional, as a consequence of the relation $a_1^2 \propto a_2^2$ [see (6)].

4. GENERALIZATION TO MULTIMODE CASE

The quadratic relations between a_i operators, (6), can be written in terms of different R-matrices. Instead of the original R-matrix (5) we can use an R-matrix of the form

$$
pR = \begin{bmatrix} 0 & 0 & 0 & R_{11,22} \\ 0 & \epsilon'' & 0 & 0 \\ 0 & 0 & \epsilon'' & 0 \\ R_{22,11} & 0 & 0 & 0 \end{bmatrix} \tag{17}
$$

where

$$R_{11,22} = R^{-1}_{22,11} = \epsilon' \frac{1 + \epsilon q}{1 - \epsilon q}$$
$$\epsilon^2 = \epsilon'^2 = \epsilon''^2 = 1 \tag{18}$$

We can reproduce the algebra in (6) from the algebra $a_i a_j - R_{ij,kl} a_l a_k = 0$ by using the R-matrix (17) and Q_{ij} [$\tilde{Q}_{im}$ from (13)]. This choice is particularly useful for generalization to multimode "peculiar" oscillators, $n > 2$.

Now we propose a generalization of the algebra (6) to n oscillators, a_i, $a_i^\dagger$, $i = 1 \ldots n$. Quadratic relations between the a_i operators are given by

$$\begin{aligned} a_i a_i &= R_{ii,jj} a_j a_j, \qquad i \neq j \\ a_i a_j &= R_{ij,ij} a_j a_i, \qquad i \neq j \end{aligned} \tag{19}$$

where no summation is assumed and where

$$\begin{aligned} R_{ii,kk} \cdot R_{jj,ii} &= R_{jj,kk}, \qquad i \neq j, \quad i \neq k, \quad j \neq k \\ R_{ii,jj} \cdot R_{jj,ii} &= 1, \qquad i \neq j \\ R_{ij,ij} &= R_{ji,ji} = \epsilon_{ij} \\ \epsilon_{ij}^2 &= 1, \qquad i \neq j \end{aligned} \tag{20}$$

and all other R-matrix elements vanish.

There are 2^{n-1} towers of states. For example, we can create them using the $a_1^\dagger$ operator under the 2^{n-1} vacua:

$$|0, \nu_2, \ldots, \nu_n\rangle, \qquad \nu_2, \ldots, \nu_n = 0, 1 \tag{21}$$

Then the algebra is completely determined by

$$a_1^\dagger a_1 = \phi_1(N_1, \nu_2, \ldots, \nu_n) \tag{22}$$

The operator $a_1^\dagger a_1$ is positive definite, i.e., the function ϕ_1 should satisfy

$$\phi_1(n_1, \nu_2, \ldots, \nu_n) > 0, \qquad \forall \nu_i = 0, 1, \quad n_1 \in \mathbf{N}_0 \tag{23}$$

The Yang–Baxter equations, associativity of the algebra, and equation (22) guarantee that the complete set of states can be written as

$$|n_1, \nu_2, \ldots, \nu_n\rangle = \frac{(a_1^\dagger)^{n_1}(a_2^\dagger)^{\nu_2}\cdots(a_n^\dagger)^{\nu_n}}{[\phi_1(n_1, \nu_2, \ldots, \nu_n)]!}|0, \ldots, 0\rangle$$
$$\nu_2, \ldots, \nu_n = 0, 1, \qquad \forall n_1 \in \mathbf{N}_0 \tag{24}$$

The actions of the a_i, $a_i^\dagger$ operators on these states are

$$a_1^\dagger|n_1, \nu_2, \ldots, \nu_n\rangle = \sqrt{\phi_1(n_1+1, \nu_2, \ldots, \nu_n)}\,|n_1+1, \nu_2, \ldots, \nu_n\rangle$$
$$a_1|n_1, \nu_2, \ldots, \nu_n\rangle = \sqrt{\phi_1(n_1, \nu_2, \ldots, \nu_n)}\,|n_1-1, \nu_2, \ldots, \nu_n\rangle$$
$$a_2^\dagger|n_1, \nu_2, \ldots, \nu_n\rangle = \sqrt{\frac{[\phi_1(n_1+2\nu_2, 1-\nu_2, \ldots, \nu_n)]!}{[\phi_1(n_1, \nu_2, \ldots, \nu_n)]!}}\,(R_{11,22})^{\nu_2}(\epsilon_{12})^{n_1} \tag{25}$$
$$\times\, |n_1+2\nu_2, 1-\nu_2, \ldots, \nu_n\rangle$$
$$a_2|n_1, \nu_2, \ldots, \nu_n\rangle = \sqrt{\frac{[\phi_1(n_1, \nu_2, \ldots, \nu_n)]!}{[\phi_1(n_1-2+2\nu_2, 1-\nu_2, \ldots, \nu_n)]!}}\,(R_{11,22})^{1-\nu_2}(\epsilon_{12})^{n_1}$$
$$\times\, |n_1-2+2\nu_2, 1-\nu_2, \ldots, \nu_n\rangle$$

and similarly for other operators a_k, $a_k^\dagger$, $k > 2$.

We define the operators $Q_{ij} = a_i a_j^\dagger$, $Q_{ij}^\dagger = a_j a_i^\dagger = Q_{ji}$, $\tilde{Q}_{ij} = a_i^\dagger a_j$, and $\tilde{Q}_{ij}^\dagger = \tilde{Q}_{ji}$. The Q_{ij}, $\tilde{Q}_{ij}$ commute with the total number operator N:

$$[N, Q_{ij}] = [N, \tilde{Q}_{ij}] = 0 \tag{26}$$

The total number operator satisfies

$$[N, a_i] = -a_i, \qquad \forall i = 1, \ldots, n \tag{27}$$
$$N|n_1, \nu_2, \ldots, \nu_n\rangle = (n_1 + \nu_2 + \cdots + \nu_n)|n_1, \nu_2, \ldots, \nu_n\rangle$$

The action of the Q_{ij} operator on the states in (24) is

$$Q_{ij}|n_1, \nu_2, \ldots, \nu_n\rangle$$
$$= a_i a_j^\dagger|n_1, \nu_2, \ldots, \nu_i, \ldots, \nu_j, \ldots, \nu_n\rangle$$
$$= \psi_{ij}(n_1, \nu_2, \ldots, \nu_n)|n_1', \nu_2, \ldots, \nu_i', \ldots, \nu_j', \ldots, \nu_n\rangle \tag{28}$$

where

$$n_1' = n_1 - 2 + 2\nu_i + 2\nu_j, \qquad i \neq j$$
$$\nu_i' = 1 - \nu_i$$
$$\nu_j' = 1 - \nu_j$$

and if $i = j$,

$$n_1' = n_1 - 1 + 2\nu_i$$
$$\nu_i' = 1 - \nu_i$$

A similar relation holds for $\tilde{Q}_{ij}$ with $\tilde{\psi}_{ij}$.

The operators a_2, $a_2^\dagger$ interconnect two towers, $|0, 0, \nu_3, \ldots, \nu_n\rangle$ and $|0, 1, \nu_3, \ldots, \nu_n\rangle$, for fixed $\nu_3, \ldots, \nu_n$ (and analogously for the operators a_k, $a_k^\dagger$, $k > 2$).

The operators Q_{ij}, Q_{ji}, $\tilde{Q}_{ij}$, $\tilde{Q}_{ji}$ braid between two towers, $|0, \ldots, \nu_i, \ldots, \nu_j, \ldots\rangle$ and $|0, \ldots, 1 - \nu_i, \ldots, 1 - \nu_j, \ldots\rangle$, for fixed ν_k, $k \neq i$, $k \neq j$, preserving the total number $N = n_1 + \nu_2 + \cdots + \nu_n$.

We point out that the states $|n_1, \nu_2, \ldots, \nu_n\rangle$, (24), are eigenstates of the operators Q_{ij}^2, $Q_{ij}Q_{ji}$, $\tilde{Q}_{ij}^2$, $\tilde{Q}_{ij}\tilde{Q}_{ji}$, which generally do not vanish. Let us define

$$\{Q_{ij}^\dagger, Q_{ij}\} = 2H_{ij} = 2H_{ji} \tag{29}$$

Then we have, analogously to the relations (15),

$$[Q_{ij}, H_{ij}] = 0, \qquad \forall i, j = 1, \ldots, n \tag{30}$$

Similar relations hold with $\tilde{Q}_{ij}$ and $\tilde{H}$ [cf. (16)].

In the limit $R_{kk,11} \to 0$, $k \geq 2$, we find one Bose oscillator a_1, $a_1^\dagger$ and $n - 1$ Fermi oscillators a_k, $a_k^\dagger$, $k \geq 2$, with $a_1^\dagger a_1 = N_1$, $Q_{ij}^2 = (a_i^\dagger a_j)^2 = 0$, $j \neq 1$ or $i \neq 1$.

5. CONCLUSION

In conclusion, we have investigated the Fock-space representation and the number operator for the "peculiar" algebra defined for a two-mode oscillator in Van der Jeugt (1993). We have shown that this algebra corresponds to the deformed supersymmetric oscillator. This deformed SUSY oscillator represents an alternative mechanism for the violation of the Pauli exclusion principle. We have proposed a simple generalization of this "peculiar" algebra to a multimode case. It is also possible to generalize the "peculiar" algebra to include arbitrary relations between powers of the operators a_i with arbitrary exponents. In this case there is no quadratic R-matrix relation.

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